

Counting r -tuples of positive integers with k -wise relatively prime components

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Abstract

Let $r \geq k \geq 2$ be fixed positive integers. Let $\varrho_{r,k}$ denote the characteristic function of the set of r -tuples of positive integers with k -wise relatively prime components, that is any k of them are relatively prime. We use the convolution method to establish an asymptotic formula for the sum $\sum_{n_1, \dots, n_r \leq x} \varrho_{r,k}(n_1, \dots, n_r)$ by elementary arguments. Our result improves the error term obtained by J. Hu [5].

Keywords: k -wise relatively prime integers; asymptotic density; multiplicative function of several variables; convolution method; error term

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1 Introduction

Let $r \geq k \geq 2$ be fixed positive integers. The positive integers n_1, \dots, n_r are called k -wise relatively prime if any k of them are relatively prime, that is $\gcd(n_{i_1}, \dots, n_{i_k}) = 1$ for every $1 \leq i_1 < \dots < i_k \leq r$. In particular, in the case $k = 2$ the integers are pairwise relatively prime and for $k = r$ they are mutually relatively prime.

Let $\mathcal{S}_{r,k}$ denote the set of r -tuples of positive integers with k -wise relatively prime components and let $\varrho_{r,k}$ stand for its characteristic function. What is the asymptotic density

$$d_{r,k} = \lim_{x \rightarrow \infty} \frac{1}{x^r} \sum_{n_1, \dots, n_r \leq x} \varrho_{r,k}(n_1, \dots, n_r)$$

of the set $\mathcal{S}_{r,k}$? Heuristically, the probability that a positive integer is divisible by a fixed prime p is $1/p$, hence the probability that given r positive integers exactly j of them are divisible by p is

$$\binom{r}{j} \frac{1}{p^j} \left(1 - \frac{1}{p}\right)^{r-j}$$

and the probability that they are k -wise relatively prime is

$$P_{r,k} = \prod_p \sum_{j=0}^{k-1} \binom{r}{j} \frac{1}{p^j} \left(1 - \frac{1}{p}\right)^{r-j}. \quad (1)$$

In the case $k = 2$ the above heuristic argumentation is given in [9, p. 55] and one has

$$P_{r,2} = \prod_p \left(1 - \frac{1}{p}\right)^{r-1} \left(1 + \frac{r-1}{p}\right). \quad (2)$$

Note that for every $r \geq k \geq 2$,

$$c \prod_{p>r-1} \left(1 - \frac{(r-1)^2}{p^2}\right) \leq P_{r,2} \leq P_{r,k} \leq P_{r,r} = \prod_p \left(1 - \frac{1}{p^r}\right),$$

with some constant $c > 0$ (depending on r), hence the infinite product (1) converges. Some approximate values of $P_{r,k}$ are shown by the next Table.

$P_{r,k}$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$
$r = 2$	0.607						
$r = 3$	0.286	0.831					
$r = 4$	0.114	0.584	0.923				
$r = 5$	0.040	0.357	0.768	0.964			
$r = 6$	0.013	0.195	0.576	0.873	0.982		
$r = 7$	0.004	0.097	0.394	0.734	0.930	0.991	
$r = 8$	0.001	0.045	0.247	0.573	0.837	0.962	0.995

Table. Approximate values of $P_{r,k}$ for $2 \leq k \leq r \leq 8$

If $k = r$, then it is well known that $d_{r,r} = P_{r,r} = 1/\zeta(r)$ is the correct value of the corresponding asymptotic density. The case $k = 2$ was treated by the author [10] proving by an inductive approach that

$$\sum_{n_1, \dots, n_r \leq x} \varrho_{r,2}(n_1, \dots, n_r) = d_{r,2} x^r + O\left(x^{r-1}(\log x)^{r-1}\right), \quad (3)$$

where $d_{r,2} = P_{r,2}$ is given by (2). Here and throughout the paper the $O(\ll)$ notation is used in the usual way, the implied constants depend only on r .

The value (2) was also deduced by J.-Y. Cai, E. Bach [1, Th. 3.3] using probabilistic arguments. P. Moree [8, Th. 2] proved (3) in the case $r = 3$ using a different approach. J. Hu [5, 6] proved that $d_{r,k} = P_{r,k}$ for every $r \geq k \geq 2$. In fact, by generalizing the method of [10] it was shown in [5] that

$$\sum_{n_1, \dots, n_r \leq x} \varrho_{r,k}(n_1, \dots, n_r) = P_{r,k} x^r + O\left(x^{r-1}(\log x)^{\delta_{r,k}}\right), \quad (4)$$

where $\delta_{r,k} = \max\left\{\binom{r-1}{j} : 1 \leq j \leq k-1\right\}$. For $k = 2$ the asymptotic formula (4) reduces to (3). We remark that the asymptotic density $d_{r,2}$ was obtained by the author [12, Sect. 7.2] by applying the generalized Wintner theorem due to N. Ushiroya [13].

Similar questions were investigated in some other recent papers. J. Hu [7] and J. A. de Reyna, R. Heyman [2] considered modified pairwise coprimality conditions and by using certain

graph representations they obtained asymptotic formulas similar to (4). Probabilistic aspects of pairwise coprimality were investigated by J. L. Fernández, P. Fernández [3]. For example, it is proved in [3] that the random variable counting the number of coprime pairs in a random sample of length r , drawn from $\{1, 2, \dots, n\}$, is asymptotically normal as r tends to infinity and $n \geq 2$ is allowed to vary with r . X. Guo, H. Xiangqian, X. Liu [4] computed the asymptotic density of the set of n -tuples of k -wise relatively prime polynomials over a finite field.

It is the goal of the present paper to use a method, which differs from all approaches mentioned above, in order to establish the asymptotic formula (4) with a better error term. More exactly, we take into account that the function $\varrho_{r,k}(n_1, \dots, n_r)$ is multiplicative, viewed as an arithmetic function of r variables. Therefore, its multiple Dirichlet series can be expressed as an Euler product and an explicit formula can be given for it. See the survey paper of the author [12] for basic properties of multiplicative functions of several variables. Then we use the convolution method to obtain the desired asymptotic formula by elementary arguments.

2 Main results

We use the notation $n = \prod_p p^{\nu_p(n)}$ for the prime power factorization of $n \in \mathbb{N}$, the product being over the primes p , where all but a finite number of the exponents $\nu_p(n)$ are zero. Furthermore, let $e_j(x_1, \dots, x_r) = \sum_{1 \leq i_1 < \dots < i_j \leq r} x_{i_1} \cdots x_{i_j}$ denote the elementary symmetric polynomials in x_1, \dots, x_r of degree j ($j \geq 0$). By convention, $e_0(x_1, \dots, x_r) = 1$.

As mentioned in the Introduction, the function $\varrho_{r,k}$ is multiplicative, which means that

$$\varrho_{r,k}(m_1 n_1, \dots, m_r n_r) = \varrho_{r,k}(m_1, \dots, m_r) \varrho_{r,k}(n_1, \dots, n_r),$$

provided that $\gcd(m_1 \cdots m_r, n_1 \cdots n_r) = 1$. Hence we have

$$\varrho_{r,k}(n_1, \dots, n_r) = \prod_p \varrho_{r,k}(p^{\nu_p(n_1)}, \dots, p^{\nu_p(n_r)})$$

for every n_1, \dots, n_r . Also, for every $\nu_1, \dots, \nu_r \geq 0$,

$$\varrho_{r,k}(p^{\nu_1}, \dots, p^{\nu_r}) = \begin{cases} 1, & \text{if there are at most } k-1 \text{ values } \nu_i \geq 1, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

For the multiple Dirichlet series of the function $\varrho_{r,k}$ we have the next result:

Theorem 2.1. *Let $r \geq k \geq 2$ and let $s_i \in \mathbb{C}$ ($1 \leq i \leq r$). If $\Re s_i > 1$ ($1 \leq i \leq r$), then*

$$\sum_{n_1, \dots, n_r=1}^{\infty} \frac{\varrho_{r,k}(n_1, \dots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}} = \zeta(s_1) \cdots \zeta(s_r) D_{r,k}(s_1, \dots, s_r),$$

where

$$D_{r,k}(s_1, \dots, s_r) = \prod_p \left(1 - \sum_{j=k}^r (-1)^{j-k} \binom{j-1}{k-1} e_j(p^{s_1}, \dots, p^{s_r}) \right)$$

is absolutely convergent if and only if $\Re(s_{i_1} + \dots + s_{i_j}) > 1$ for every $1 \leq i_1 < \dots < i_j \leq r$ with $k \leq j \leq r$.

In the case $k = 2$, Theorem 2.1 was deduced by the author [12, Sect. 5.1], based on an identity concerning a generalization of the Busche-Ramanujan identity. See [11, Eq. (4.2)].

We prove the following asymptotic formula:

Theorem 2.2. *If $r \geq k \geq 2$, then*

$$\sum_{n_1, \dots, n_r \leq x} \varrho_{r,k}(n_1, \dots, n_r) = A_{r,k} x^r + O(R_{r,k}(x)),$$

where

$$A_{r,k} = \prod_p \left(1 - \sum_{j=k}^r (-1)^{j-k} \binom{r}{j} \binom{j-1}{k-1} \frac{1}{p^j} \right) \quad (6)$$

and

$$R_{r,k}(x) = \begin{cases} x^{r-1}, & \text{if } r \geq k \geq 3, \\ x^{r-1}(\log x)^{r-1}, & \text{if } r \geq k = 2. \end{cases} \quad (7)$$

For $k \geq 3$ the error term $R_{r,k}(x)$ is better than in (4), obtained by J. Hu [5]. Note also that $A_{r,k} = P_{r,k}$, given by (1), which follows by some simple properties of the binomial coefficients.

3 Preliminaries

Consider the polynomial

$$f(x) = \prod_{j=1}^r (x - x_j) = \sum_{j=0}^r (-1)^j e_j(x_1, \dots, x_r) x^{r-j}. \quad (8)$$

We will use that its m -th derivative is

$$f^{(m)}(x) = m! \sum_{j=0}^r (-1)^j \binom{r-j}{m} e_j(x_1, \dots, x_r) x^{r-j-m} \quad (0 \leq m \leq r), \quad (9)$$

and on the other hand

$$f^{(m)}(x) = m! \sum_{1 \leq i_1 < \dots < i_m \leq r} \prod_{\substack{j=1 \\ j \neq i_1, \dots, i_m}}^r (x - x_j) \quad (0 \leq m \leq r). \quad (10)$$

We also need the following auxiliary results:

Lemma 3.1. *If $a_j \in \mathbb{C}$ ($1 \leq j \leq r$), then*

$$\prod_{j=1}^r a_j = \sum_{\ell=0}^r (-1)^\ell \sum_{1 \leq i_1 < \dots < i_\ell \leq r} (1 - a_{i_1}) \cdots (1 - a_{i_\ell}),$$

where the term for $\ell = 0$ is considered to be 1.

Proof. Follows from (8) by putting $x = 1$ and $x_j = 1 - a_j$ ($1 \leq j \leq r$). \square

Lemma 3.2. *We have the polynomial identity*

$$\sum_{j=0}^{k-1} \sum_{1 \leq i_1 < \dots < i_j \leq r} x_{i_1} \cdots x_{i_j} \prod_{\substack{\ell=1 \\ \ell \neq i_1, \dots, i_j}}^r (1 - x_\ell) = 1 - \sum_{j=k}^r (-1)^{j-k} \binom{j-1}{k-1} e_j(x_1, \dots, x_r), \quad (11)$$

where on the left hand side the term for $j = 0$ is considered to be $\prod_{\ell=1}^r (1 - x_\ell)$.

Note that the left hand side of (11) is a symmetric polynomial in x_1, \dots, x_r and the right hand side shows how it can be written as a polynomial of the elementary symmetric polynomials.

Proof. By using Lemma 3.1,

$$\begin{aligned} S &:= \sum_{j=0}^{k-1} \sum_{1 \leq i_1 < \dots < i_j \leq r} x_{i_1} \cdots x_{i_j} \prod_{\substack{\ell=1 \\ \ell \neq i_1, \dots, i_j}}^r (1 - x_\ell) \\ &= \sum_{j=0}^{k-1} \sum_{1 \leq i_1 < \dots < i_j \leq r} \sum_{m=0}^r (-1)^m \sum_{1 \leq t_1 < \dots < t_m \leq r} (1 - x_{i_{t_1}}) \cdots (1 - x_{i_{t_m}}) \prod_{\substack{\ell=1 \\ \ell \neq i_1, \dots, i_j}}^r (1 - x_\ell). \end{aligned}$$

In the last product a number of j factors are missing from the factors $1 - x_1, \dots, 1 - x_r$. But a number of m factors from the missing ones are present in front of the last product. Hence the number of missing factors is $q = j - m$, where $0 \leq q \leq j$. We obtain

$$\begin{aligned} S &= \sum_{j=0}^{k-1} \sum_{1 \leq i_1 < \dots < i_j \leq r} \sum_{q=0}^j (-1)^{j-q} \sum_{1 \leq u_1 < \dots < u_q \leq j} \prod_{\substack{\ell=1 \\ \ell \neq i_{u_1}, \dots, i_{u_q}}}^r (1 - x_\ell) \\ &= \sum_{j=0}^{k-1} (-1)^j \sum_{q=0}^j (-1)^q \sum_{1 \leq u_1 < \dots < u_q \leq j} \sum_{1 \leq i_1 < \dots < i_j \leq r} \prod_{\substack{\ell=1 \\ \ell \neq i_{u_1}, \dots, i_{u_q}}}^r (1 - x_\ell), \end{aligned}$$

by regrouping the terms. Here for fixed u_1, \dots, u_q the values $a_1 = i_{u_1}, \dots, a_q = i_{u_q}$ are also fixed and the other $r - q$ values of i_1, \dots, i_j can be selected in $\binom{r-q}{j-q}$ ways. Therefore,

$$\begin{aligned} S &= \sum_{j=0}^{k-1} (-1)^j \sum_{q=0}^j (-1)^q \binom{r-q}{j-q} \sum_{1 \leq a_1 < \dots < a_q \leq r} \prod_{\substack{\ell=1 \\ \ell \neq a_1, \dots, a_q}}^r (1 - x_\ell) \\ &= \sum_{q=0}^{k-1} (-1)^q \sum_{1 \leq a_1 < \dots < a_q \leq r} \left(\prod_{\substack{\ell=1 \\ \ell \neq a_1, \dots, a_q}}^r (1 - x_\ell) \right) \sum_{j=q}^{k-1} (-1)^j \binom{r-q}{j-q}, \end{aligned}$$

where the last sum is $(-1)^{k-1} \binom{r-q-1}{k-q-1}$ and we deduce

$$S = \sum_{q=0}^{k-1} (-1)^{q+k-1} \binom{r-q-1}{k-q-1} \sum_{1 \leq a_1 < \dots < a_q \leq r} \prod_{\substack{\ell=1 \\ \ell \neq a_1, \dots, a_q}}^r (1 - x_\ell).$$

Now using the identities (10) and (9) for $x = 1$ we conclude

$$\begin{aligned} S &= \sum_{q=0}^{k-1} (-1)^{q+k-1} \binom{r-q-1}{k-q-1} \frac{1}{q!} f^{(q)}(1) \\ &= \sum_{q=0}^{k-1} (-1)^{q+k-1} \binom{r-q-1}{k-q-1} \sum_{j=0}^r (-1)^j \binom{r-j}{q} e_j(x_1, \dots, x_r) \\ &= \sum_{j=0}^r (-1)^{j-k+1} e_j(x_1, \dots, x_r) \sum_{q=0}^{k-1} (-1)^q \binom{r-q-1}{(k-1)-q} \binom{r-j}{q}, \end{aligned}$$

where the last sum is $\binom{j-1}{k-1}$ by the Vandermonde identity. Hence

$$S = \sum_{j=0}^r (-1)^{j-k+1} e_j(x_1, \dots, x_r) \binom{j-1}{k-1} = 1 - \sum_{j=k}^r (-1)^{j-k} \binom{j-1}{k-1} e_j(x_1, \dots, x_r),$$

which completes the proof. \square

4 Proofs

Proof of Theorem 2.1. The function $(n_1, \dots, n_r) \mapsto \varrho_{r,k}(n_1, \dots, n_r)$ is multiplicative, hence its Dirichlet series can be expanded into an Euler product. Using (5) we deduce

$$\begin{aligned} \sum_{n_1, \dots, n_r=1}^{\infty} \frac{\varrho_{r,k}(n_1, \dots, n_r)}{n_1^{s_1} \dots n_r^{s_r}} &= \prod_p \sum_{\nu_1, \dots, \nu_r=0}^{\infty} \frac{\varrho_{r,k}(p^{\nu_1}, \dots, p^{\nu_r})}{p^{\nu_1 s_1 + \dots + \nu_r s_r}} \\ &= \prod_p \left(1 + \sum_{j=1}^{k-1} \sum_{1 \leq i_1 < \dots < i_j \leq r} \sum_{\nu_{i_1}, \dots, \nu_{i_j}=1}^{\infty} \frac{1}{p^{\nu_{i_1} s_{i_1} + \dots + \nu_{i_j} s_{i_j}}} \right) \\ &= \prod_p \left(1 + \sum_{j=1}^{k-1} \sum_{1 \leq i_1 < \dots < i_j \leq r} \frac{1}{p^{s_{i_1}}} \left(1 - \frac{1}{p^{s_{i_1}}} \right)^{-1} \dots \frac{1}{p^{s_{i_j}}} \left(1 - \frac{1}{p^{s_{i_j}}} \right)^{-1} \right) \\ &= \zeta(s_1) \dots \zeta(s_r) \prod_p \left(\prod_{\ell=1}^r \left(1 - \frac{1}{p^{s_\ell}} \right) + \sum_{j=1}^{k-1} \sum_{1 \leq i_1 < \dots < i_j \leq r} \frac{1}{p^{s_{i_1} + \dots + s_{i_j}}} \prod_{\substack{\ell=1 \\ \ell \neq i_1, \dots, i_j}}^r \left(1 - \frac{1}{p^{s_\ell}} \right) \right) \end{aligned}$$

$$= \zeta(s_1) \cdots \zeta(s_r) \prod_p \left(1 - \sum_{j=k}^r (-1)^{j-k} \binom{j-1}{k-1} e_j(p^{s_1}, \dots, p^{s_r}) \right),$$

by using Lemma 3.2 for $x_1 = p^{s_1}, \dots, x_r = p^{s_r}$ in the last step. \square

Proof of Theorem 2.2. According to Theorem 2.1, for every $n_1, \dots, n_r \in \mathbb{N}$,

$$\varrho_{r,k}(n_1, \dots, n_r) = \sum_{d_1 | n_1, \dots, d_r | n_r} \psi_{r,k}(d_1, \dots, d_r), \quad (12)$$

where

$$\sum_{n_1, \dots, n_r=1}^{\infty} \frac{\psi_{r,k}(n_1, \dots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}} = D_{r,k}(s_1, \dots, s_r).$$

The function $\psi_{r,k}$ is also multiplicative, symmetric in the variables and for any prime powers $p^{\nu_1}, \dots, p^{\nu_r}$,

$$\psi_{r,k}(p^{\nu_1}, \dots, p^{\nu_r}) = \begin{cases} 1, & \nu_1 = \dots = \nu_r = 0, \\ (-1)^{j-k+1} \binom{j-1}{k-1}, & \nu_1, \dots, \nu_r \in \{0, 1\}, \quad j := \nu_1 + \dots + \nu_r \geq k, \\ 0, & \text{otherwise.} \end{cases} \quad (13)$$

Note that $\psi_{r,k}(p^{\nu_1}, \dots, p^{\nu_r}) = 0$ provided that $\nu_i \geq 2$ for at least one $1 \leq i \leq r$, or $\nu_1, \dots, \nu_r \in \{0, 1\}$ and $\nu_1 + \dots + \nu_r < k$. For $k \geq 2$ one has $\psi_{r,k}(p, 1, \dots, 1) = 0$ and for $k \geq 3$ one has $\psi_{r,k}(p, p, 1, \dots, 1) = 0$, where p is any prime.

From (12) we deduce

$$\begin{aligned} \sum_{n_1, \dots, n_r \leq x} \varrho_{r,k}(n_1, \dots, n_r) &= \sum_{d_1, \dots, d_r \leq x} \psi_{r,k}(d_1, \dots, d_r) \left\lfloor \frac{x}{d_1} \right\rfloor \cdots \left\lfloor \frac{x}{d_r} \right\rfloor \\ &= \sum_{d_1, \dots, d_r \leq x} \psi(d_1, \dots, d_r) \left(\frac{x}{d_1} + O(1) \right) \cdots \left(\frac{x}{d_r} + O(1) \right) \\ &= x^r \sum_{d_1, \dots, d_r \leq x} \frac{\psi_{r,k}(d_1, \dots, d_r)}{d_1 \cdots d_r} + Q_{r,k}(x), \end{aligned} \quad (14)$$

with

$$Q_{r,k}(x) \ll \sum_{u_1, \dots, u_r} x^{u_1 + \dots + u_r} \sum_{d_1, \dots, d_r \leq x} \frac{|\psi_{r,k}(d_1, \dots, d_r)|}{d_1^{u_1} \cdots d_r^{u_r}},$$

where the first sum is over $u_1, \dots, u_r \in \{0, 1\}$ such that at least one u_i is 0. Let u_1, \dots, u_r be fixed and assume that $u_r = 0$. Since $(x/d_i)^{u_i} \leq x/d_i$ for every i , we have

$$A := x^{u_1 + \dots + u_r} \sum_{d_1, \dots, d_r \leq x} \frac{|\psi_{r,k}(d_1, \dots, d_r)|}{d_1^{u_1} \cdots d_r^{u_r}} \leq x^{r-1} \sum_{d_1, \dots, d_r \leq x} \frac{|\psi_{r,k}(d_1, \dots, d_r)|}{d_1 \cdots d_{r-1}}$$

Assume that $k \geq 3$. Then

$$A \leq x^{r-1} \sum_{d_1, \dots, d_r=1}^{\infty} \frac{|\psi_{r,k}(d_1, \dots, d_r)|}{d_1 \cdots d_{r-1}} \ll x^{r-1},$$

since the series $D_{r,k}(1, \dots, 1, 0)$ is absolutely convergent for $k \geq 3$ by Theorem 2.1. We obtain that

$$Q_{r,k}(x) \ll x^{r-1} \quad (k \geq 3). \quad (15)$$

If $k = 2$, then

$$\begin{aligned} A &\leq x^{r-1} \prod_{p \leq x} \sum_{\nu_1, \dots, \nu_r=0}^{\infty} \frac{|\psi_{r,2}(p^{\nu_1}, \dots, p^{\nu_r})|}{p^{\nu_1 + \dots + \nu_{r-1}}} \\ &= x^{r-1} \prod_{p \leq x} \left(1 + \frac{r-1}{p} + \frac{c_2}{p^2} + \dots + \frac{c_{r-1}}{p^{r-1}} \right), \end{aligned} \quad (16)$$

by (13), where c_2, \dots, c_{r-1} are certain positive integers, using also that we have p in the denominator if and only if $\nu_r = 1$ and exactly one of ν_1, \dots, ν_{r-1} is 1, the rest being 0, which occurs $r-1$ times. We deduce that

$$A \ll x^{r-1} \prod_{p \leq x} \left(1 + \frac{1}{p} \right)^{r-1} \ll x^{r-1} (\log x)^{r-1}$$

by Mertens' theorem. This shows that

$$Q_{r,2}(x) \ll x^{r-1} (\log x)^{r-1}. \quad (17)$$

Furthermore, for the main term of (14) we have

$$\begin{aligned} &\sum_{d_1, \dots, d_r \leq x} \frac{\psi_{r,k}(d_1, \dots, d_r)}{d_1 \cdots d_r} \\ &= \sum_{d_1, \dots, d_r=1}^{\infty} \frac{\psi_{r,k}(d_1, \dots, d_r)}{d_1 \cdots d_r} - \sum_{\emptyset \neq I \subseteq \{1, \dots, r\}} \sum_{\substack{d_i > x, i \in I \\ d_j \leq x, j \notin I}} \frac{\psi_{r,k}(d_1, \dots, d_r)}{d_1 \cdots d_r}, \end{aligned} \quad (18)$$

where the series is convergent by Theorem 2.1 and its sum is $D_{r,k}(1, \dots, 1) = A_{r,k}$, given by (6).

Let I be fixed and assume that $I = \{1, 2, \dots, t\}$, that is $d_1, \dots, d_t > x$ and $d_{t+1}, \dots, d_r \leq x$, where $t \geq 1$. We estimate the sum

$$B := \sum_{\substack{d_1, \dots, d_t > x \\ d_{t+1}, \dots, d_r \leq x}} \frac{|\psi_{r,k}(d_1, \dots, d_r)|}{d_1 \cdots d_r}$$

by distinguishing the following cases:

Case i) $k \geq 3, t \geq 1$:

$$B < \frac{1}{x} \sum_{d_1, \dots, d_r=1}^{\infty} \frac{|\psi_{r,k}(d_1, \dots, d_r)|}{d_2 \cdots d_r} \ll \frac{1}{x},$$

since the series is convergent by Theorem 2.1.

Case ii) $k = 2, t \geq 3$: if $0 < \varepsilon < 1/2$, then

$$\begin{aligned} B &= \sum_{\substack{d_1, \dots, d_t > x \\ d_{t+1}, \dots, d_r \leq x}} \frac{|\psi_{r,2}(d_1, \dots, d_r)| d_1^{\varepsilon-1/2} \cdots d_t^{\varepsilon-1/2}}{d_1^{1/2+\varepsilon} \cdots d_t^{1/2+\varepsilon} d_{t+1} \cdots d_r} \\ &< x^{t(\varepsilon-1/2)} \sum_{d_1, \dots, d_r=1}^{\infty} \frac{|\psi_{r,2}(d_1, \dots, d_r)|}{d_1^{1/2+\varepsilon} \cdots d_t^{1/2+\varepsilon} d_{t+1} \cdots d_r} \ll x^{t(\varepsilon-1/2)}, \end{aligned}$$

since the series is convergent (for $t \geq 1$). Using that $t(\varepsilon - 1/2) < -1$ for $0 < \varepsilon < (t-2)/(2t)$, here we need $t \geq 3$, we obtain $B \ll \frac{1}{x}$.

Case iii) $k = 2, t = 1$: Let $d_1 > x, d_2, \dots, d_r \leq x$ and consider a prime p . If $p \mid d_i$ for an $i \in \{2, \dots, r\}$, then $p \leq x$. If $p \mid d_1$ and $p > x$, then $p \nmid d_i$ for every $i \in \{2, \dots, r\}$ and $\psi_{r,2}(d_1, \dots, d_r) = 0$ by its definition (13). Hence it is enough to consider the primes $p \leq x$. We deduce

$$\begin{aligned} B &< \frac{1}{x} \sum_{\substack{d_1 > x \\ d_2, \dots, d_r \leq x}} \frac{|\psi_{r,2}(d_1, \dots, d_r)|}{d_2 \cdots d_r} \\ &\leq \frac{1}{x} \prod_{p \leq x} \sum_{\nu_1, \dots, \nu_r=0}^{\infty} \frac{|\psi_{r,2}(p^{\nu_1}, \dots, p^{\nu_r})|}{p^{\nu_2 + \cdots + \nu_r}} \ll \frac{1}{x} (\log x)^{r-1}, \end{aligned}$$

similar to the estimate of (16).

Case iv) $k = 2, t = 2$: We split the sum B into two sums, namely

$$\begin{aligned} B &= \sum_{\substack{d_1 > x, d_2 > x \\ d_3, \dots, d_r \leq x}} \frac{|\psi_{r,2}(d_1, \dots, d_r)|}{d_1 \cdots d_r} \\ &= \sum_{\substack{d_1 > x^{3/2}, d_2 > x \\ d_3, \dots, d_r \leq x}} \frac{|\psi_{r,2}(d_1, \dots, d_r)|}{d_1 \cdots d_r} + \sum_{\substack{x^{3/2} \geq d_1 > x, d_2 > x \\ d_3, \dots, d_r \leq x}} \frac{|\psi_{r,2}(d_1, \dots, d_r)|}{d_1 \cdots d_r} =: B_1 + B_2, \end{aligned}$$

say, where

$$\begin{aligned} B_1 &= \sum_{\substack{d_1 > x^{3/2}, d_2 > x \\ d_3, \dots, d_r \leq x}} \frac{|\psi_{r,2}(d_1, \dots, d_r)|}{d_1^{1/3} d_2 \cdots d_r} \frac{1}{d_1^{2/3}} \\ &< \frac{1}{x} \sum_{d_1, \dots, d_r=1}^{\infty} \frac{|\psi_{r,2}(d_1, \dots, d_r)|}{d_1^{1/3} d_2 \cdots d_r} \ll \frac{1}{x}, \end{aligned}$$

since the series is convergent. Furthermore,

$$B_2 < \frac{1}{x} \sum_{\substack{x^{3/2} \geq d_1, d_2 > x \\ d_3, \dots, d_r \leq x}} \frac{|\psi_{r,2}(d_1, \dots, d_r)|}{d_1 d_3 \cdots d_r},$$

where $d_1 \leq x^{3/2}$, $d_2 > x$, $d_3, \dots, d_r \leq x$. Consider a prime p . If $p \mid d_i$ for an $i \in \{1, 3, \dots, r\}$, then $p \leq x^{3/2}$. If $p \mid d_2$ and $p > x^{3/2}$, then $p \nmid d_i$ for every $i \in \{1, 3, \dots, r\}$ and $\psi_{r,2}(d_1, \dots, d_r) = 0$ by its definition. Hence it is enough to consider the primes $p \leq x^{3/2}$. We deduce, cf. the estimate of (16),

$$B_2 < \frac{1}{x} \prod_{p \leq x^{3/2}} \sum_{\nu_1, \dots, \nu_r=0}^{\infty} \frac{|\psi_{r,2}(p^{\nu_1}, \dots, p^{\nu_r})|}{p^{\nu_1 + \nu_3 + \dots + \nu_r}} \ll \frac{1}{x} (\log x^{3/2})^{r-1} \ll \frac{1}{x} (\log x)^{r-1}.$$

Hence, given any $t \geq 1$, we have $B \ll \frac{1}{x}$ for $k \geq 3$ and $B \ll \frac{1}{x} (\log x)^{r-1}$ for $k = 2$. Therefore, by (18),

$$\sum_{d_1, \dots, d_r \leq x} \frac{\psi_{r,k}(d_1, \dots, d_r)}{d_1 \cdots d_r} = A_{r,k} + O(R_{r,k}(x)) \quad (19)$$

with the notation (6) and (7).

The proof is complete by putting together (14), (15), (17) and (19). \square

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